Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 2, pp. 337–342, 1996

0021–8928/96 $24.00+0.00

S0021-8928(96)00044-5

FORCED FLEXURAL OSCILLATIONS OF AN ELASTIC HALF-STRIP FOR MIXED BOUNDARY CONDITIONS†

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(*Received* 30 *January* 1995)

The steady flexural oscillations of an elastic half-strip with stress-free longitudinal boundaries and kinematic loading specified at the end are considered. The solution reduces to a representation within the framework of the method of homogeneous solutions, describing the stress–strain state both in the inner region of the half-strip and which adequately reflects the features of the stressed state at the corner points. It is shown analytically and numerically that the infinite algebraic systems for determining the coefficients of the expansion are solvable. © 1996 Elsevier Science Ltd. All rights reserved.

The solution of such problems by alternative methods was considered previously in [1–3].

Suppose that, in a dimensionless system of coordinates, the half-strip occupies the region $0 \leqslant x < \infty$, $|y| \leqslant 1$, and the following conditions are satisfied on the boundaries of the region

$$\sigma_y(x,\pm 1) = \tau_{xy}(x,\pm 1) = 0$$

$$u(0,y) = f(y)\exp(-i\omega t), \quad v(0,y) = g(y)\exp(-i\omega t)$$
$$f(-y) = -f(y), \quad g(-y) = g(y)$$

where the functions $f(y)$ and $g(y)$ are continuously differentiable.

The material of the half-strip has a shear modulus G, a Poisson's ratio ν and a density ρ. The time factor $\exp(-i\omega t)$ will henceforth be omitted.

The initial problem can be represented in the form of the sum of two problems, for which $f(y)$ and $g(y)$ are alternately assumed to be zero.

The solution of the first problem is constructed by superposition of the solutions for an infinite strip $|x| \leqslant x, < \infty$, $|y| \leqslant 1$ with the condition of symmetry about the y axis, and for a half-plane $x \geqslant 0$ $|y| < \infty$ with boundary conditions $u(0, y) = 0$, $\tau_{xy}(0, y) = 2G\tau(y)$. An integral Fourier transformation is used when constructing the solution, and the radiation conditions for each type of wave are taken into account [4]. As a result we obtain (the integration with respect to λ and α is carried out from zero to infinity)

$$u(x,y) = \frac{2}{\pi}\int [c_1(\lambda)\operatorname{sh} l_1 y + c_2(\lambda)\operatorname{sh} l_2 y]\sin\lambda x d\lambda +$$
$$+\frac{2}{\pi}\int c(\alpha)(\exp k_1 x - \exp k_2 x)\sin\alpha y d\alpha$$
$$v(x,y) = -\frac{2}{\pi}\int \left[c_1(\lambda)\frac{l_1}{\lambda}\operatorname{ch} l_1 y + c_2(\lambda)\frac{\lambda}{l_2}\operatorname{ch} l_2 y\right]\cos\lambda x d\lambda +$$
$$+\frac{2}{\pi}\int c(\alpha)\left(\frac{\alpha}{k_1}\exp k_1 x - \frac{k_2}{\alpha}\exp k_2 x\right)\cos\alpha y d\alpha$$
$$\frac{\sigma_x(x,y)}{2G} = \frac{2}{\pi}\int \left[c_1(\lambda)\frac{2l_1^2 + \varkappa_2^2}{2\lambda}\operatorname{sh} l_1 y + c_2(\lambda)\lambda \operatorname{sh} l_2 y\right]\cos\lambda x d\lambda +$$
$$+\frac{2}{\pi}\int c(\alpha)\left(\frac{\alpha^2 + k_2^2}{2k_1}\exp k_1 x - k_2\exp k_2 x\right)\sin\alpha y d\alpha$$

†*Prikl. Mat. Mekh.* Vol. 60, No. 2, pp. 341–345, 1996.

$$\frac{\sigma_y(x,y)}{2G} = -\frac{2}{\pi}\int\left[c_1(\lambda)\frac{\lambda^2+l_2^2}{2\lambda}\operatorname{sh} l_1 y + c_2(\lambda)\lambda \operatorname{sh} l_2 y\right]\cos\lambda x d\lambda - \tag{1}$$

$$-\frac{2}{\pi}\int c(\alpha)\left(\frac{2k_1^2+\varkappa_2^2}{2k_1}\exp k_1 x - k_2\exp k_2 x\right)\sin\alpha y d\alpha$$

$$\frac{\tau_{xy}(x,y)}{2G} = \frac{2}{\pi}\int\left[c_1(\lambda)l_1 \operatorname{ch} l_1 y + c_2(\lambda)\frac{\lambda^2+l_2^2}{2l_2}\operatorname{ch} l_2 y\right]\sin\lambda x d\lambda +$$

$$+\frac{2}{\pi}\int c(\alpha)\left(\alpha\exp k_1 x - \frac{\alpha^2+k_2^2}{2\alpha}\exp k_2 x\right)\cos\alpha y d\alpha$$

$$l_j = \begin{cases} i\sqrt{\varkappa_j^2-\lambda^2}, & \lambda<\varkappa_j \\ -\sqrt{\lambda^2-\varkappa_j^2}, & \lambda>\varkappa_j \end{cases}, \quad k_j = \begin{cases} i\sqrt{\varkappa_j^2-\alpha^2}, & \alpha<\varkappa_j \\ -\sqrt{\alpha^2-\varkappa_j^2}, & \alpha>\varkappa_j \end{cases}, \quad j=1,2$$

$$\varkappa_1^2 = \frac{1-2\nu}{2(1-\nu)}\varkappa_2^2, \quad \varkappa_2^2 = \frac{\rho}{G}\omega^2$$

Here $c_j(\lambda)$ and $c(\alpha)$ are unknown functions, determined when the boundary conditions are satisfied. The solution constructed satisfies the antisymmetry conditions $u(x,-y) = -y(x,y)$, $\upsilon(x,-y) = \upsilon(x,y)$, and also the boundary condition $u(0,y) = 0$.

We will represent the solution (1) in the form of series in the homogeneous solutions for flexural oscillations of an infinite strip with free boundaries and we will establish a relationship between the coefficients of the expansion and the stresses at the end of the half-strip.

We satisfy the conditions $\sigma_y(x,1) = \tau_{xy}(x,1) = 0$, writing them in the form

$$\frac{\lambda^2+l_2^2}{2\lambda}\operatorname{sh} l_1 c_1(\lambda) + \lambda\operatorname{sh} l_2 c_2(\lambda) = D_1(\lambda)$$

$$l_1\operatorname{ch} l_1 c_1(\lambda) + \frac{\lambda^2+l_2^2}{2l_2}\operatorname{ch} l_2 c_2(\lambda) = D_2(\lambda) \tag{2}$$

Here

$$D_1(\lambda) = \frac{2}{\pi}\int c(\alpha)\left[\frac{2k_1^2+\varkappa_2^2}{2(k_1^2+\lambda^2)} - \frac{k_2^2}{k_2^2+\lambda^2}\right]\sin\alpha d\alpha$$

$$D_2(\lambda) = -\frac{2\lambda}{\pi}\int c(\alpha)\left[\frac{\alpha}{k_1^2+\lambda^2} - \frac{\alpha^2+k_2^2}{2\alpha(k_2^2+\lambda^2)}\right]\cos\alpha d\alpha \tag{3}$$

The solution of system (2) has the form

$$c_j(\lambda) = \Delta_j(\lambda)/\Delta(\lambda)$$

$$\Delta_1(\lambda) = 2\lambda\left[\left(\lambda^2+l_2^2\right)\operatorname{ch} l_2 D_1(\lambda) - 2\lambda l_2\operatorname{sh} l_2 D_2(\lambda)\right]$$

$$\Delta_2(\lambda) = -2l_2\left[2\lambda l_1\operatorname{ch} l_1 D_1(\lambda) - \left(\lambda^2+l_2^2\right)\operatorname{sh} l_1 D_2(\lambda)\right]$$

$$\Delta(\lambda) = \left(\lambda^2+l_2^2\right)^2\operatorname{sh} l_1\operatorname{ch} l_2 - 4\lambda^2 l_1 l_2\operatorname{sh} l_2\operatorname{ch} l_1$$

This representation, using relations (3), enables us to express $c_j(\lambda)$ in terms of $c(\alpha)$ and to convert expressions (1) by expanding the integrals in them in terms of residues in accordance with the principle of limiting absorption [5]. As a result we obtain (the summation over k is from unity to infinity)

$$u(x,y)=\sum P_k u_k(y)\exp(i\lambda_k x),\quad \upsilon(x,y)=\sum P_k \upsilon_k(y)\exp(i\lambda_k x),\quad x\geqslant 0$$

$$\frac{\sigma_x(x,y)}{2G}=\sum P_k\sigma_k(y)\exp(i\lambda_k x),\quad \frac{\tau_{xy}(x,y)}{2G}=\sum P_k\tau_k(y)\exp(i\lambda_k x),\quad x>0$$

$$u_k(y)=i(\operatorname{sh} l_{1k}y+Q_k\operatorname{sh} l_{2k}y),\quad \upsilon_k(y)=\frac{l_{1k}}{\lambda_k}\operatorname{ch} l_{1k}y+\frac{\lambda_k}{l_{2k}}Q_k\operatorname{ch} l_{2k}y$$

$$\sigma_k(y)=-\left[\frac{2l_{1k}^2+\varkappa_2^2}{2\lambda_k}\operatorname{sh} l_{1k}y+\lambda_k Q_k\operatorname{sh} l_{2k}y\right] \tag{4}$$

$$\tau_k(y)=i\left[l_{1k}\operatorname{ch} l_{1k}y+\frac{\lambda_k^2+l_{2k}^2}{2l_{2k}}Q_k\operatorname{ch} l_{2k}y\right]$$

$$P_k=-2i\frac{\Delta_1(\lambda_k)}{\Delta'(\lambda_k)},\quad Q_k=\frac{-2l_{1k}l_{2k}}{\lambda_k^2+l_{2k}^2}\frac{\operatorname{ch} l_{1k}}{\operatorname{ch} l_{2k}}$$

Here λ_k are the roots of the equation $\Delta(\lambda)=0$, which lie in the upper half-plane or on the positive part of the real axis.

The functions $u_k(y)$,$\upsilon_k(y)$, $\sigma_k(y)$, $\tau_k(y)$ are homogeneous solutions for the flexural oscillations of an infinite strip with free boundaries.

We will obtain the relation between the coefficients P_k and the shear stresses at the end $\tau(y)$.

Transforming the integral in the expression for $\tau_{xy}(0,y)$ from (1) we obtain (the integration with respect to t is carried out from zero to unity)

$$c(\alpha)=\frac{2\alpha}{\varkappa_2^2}\int\tau(t)\cos\alpha t dt$$

Then

$$D_1(\lambda)=-\frac{1}{\varkappa_2^2}\int\left[\left(\lambda^2+l_2^2\right)\exp l_1\operatorname{ch} l_1 t-2\lambda^2\exp l_2\operatorname{ch} l_2 t\right]\tau(t)dt$$

$$D_2(\lambda)=-\frac{\lambda}{\varkappa_2^2}\int\left(2l_1\exp l_1\operatorname{ch} l_1 t-\frac{\lambda^2+l_2^2}{l_2}\exp l_2\operatorname{ch} l_2 t\right)\tau(t)dt$$

Using the expressions obtained and the properties of the roots λ_k, we convert the functions $\Delta_1(\lambda_k)$ and reduce the coefficients P_k to the final form

$$P_k=-2w_k^{-1}\int\tau(t)\upsilon_k(t)dt,\quad w_k=i\varkappa_2^2\frac{\Delta'(\lambda_k)}{8\lambda^4 l_{2k}}\frac{\operatorname{sh} l_{1k}}{\operatorname{sh} l_{2k}} \tag{5}$$

This representation is similar in form to that obtained in [6] for the problem of the forced oscillations of a half-strip with crossed boundary conditions at the end.

The expressions given in (4) for the displacements are convergent series and hold both inside the half-strip and over the whole end. The representation of the stresses in the form of series holds when $x>0$, since when $x=0$ these series diverge in a finite neighbourhood of the corner points $y=\pm 1$ [1].

Following the method described previously in [7, 8], we will seek the stresses at the end of the half-strip in the class of functions of the form

$$S(y)=\frac{H(y)}{(1-y)^\gamma(1+y)^\gamma},\quad 0<\gamma<1$$

where $H(y)$ is a function which satisfies the Hölder condition for $|y|\leqslant 1$. Then, taking the evenness and oddness properties of the stresses τ_{xy} and σ_x into account we obtain the representation

$$\tau(y)=\frac{C_1}{(1-y^2)^\gamma}+\tau_1(y),\quad \sigma(y)=\frac{C_2 y}{(1-y^2)^\gamma}+\sigma_1(y),\quad \sigma(y)=\frac{\sigma_x(0,y)}{2G} \tag{6}$$

where $\tau_1(y)$ and $\sigma_1(y)$ are continuous functions when $|y| \leqslant 1$, C_1, C_2 are unknown constants, and the quantity γ is given by the equation

$$(3-4\nu)\sin^2(\pi\gamma/2)=(1-\gamma)^2-(1-2\nu)^2,\quad 0<\gamma<\tfrac{1}{2}$$

Substituting the expressions for $\tau(y)$ from (6) into (5) we obtain the relation

$$C_1 b_k+B_k=P_k,\quad b_k=-2w_k^{-1}\int \upsilon_k(t)(1-t^2)^{-\gamma}dt,\quad B_k=-2w_k^{-1}\int \tau_1(t)\upsilon_k(t)dt \tag{7}$$

which is used to obtain the relation between the stress fields in the inner region of the half-strip and at the end.

The following representation for the shear stress at the end follows from (1) and (6)

$$\tau_{xy}(0,y)/(2G)=C_1(1-y^2)^{-\gamma}+\sum B_k\tau_k(y) \tag{8}$$

In order to obtain an expression for the normal stress, we will use Betti's theorem and represent (5) in the form

$$P_k=2w_k^{-1}\int\left[\sigma(t)u_k(t)-g(t)\tau_k(t)\right]dt$$

Then, like (8) we obtain

$$\sigma_x(0,y)/(2G)=C_2 y(1-y^2)^{-\gamma}+\sum\left(B_k+C_1 b_k-C_2 a_k\right)\sigma_k(y) \tag{9}$$

$$a_k=2w_k^{-1}\int t u_k(t)(1-t^2)^{-\gamma}dt$$

Using asymptotic estimates as $k\to\infty$, we can establish that the series in (8) and (9) converge over the whole end of the half-strip [2].

Hence, the problem reduces to determining the unknown constants P_k, B_k, C_1, C_2.

To obtain the unknown P_k we will use Reissner's variational principle, which leads to the following infinite algebraic system

$$\sum a_{nk}P_k=(g,\tau_n),\quad n=1,2,\ldots$$

$$a_{nk}=(\sigma_n,u_k)+(\tau_n,\upsilon_k),\quad (f_1,f_2)=\int f_1(t)f_2(t)dt$$

An analysis of the asymptotic behaviour of the coefficients of the system, based on well-known criteria [9], shows that its solution exists and can be obtained by reduction.

To determine the unknown constants C_1 and C_2 we use the conditions

$$\sum(B_k+C_1 b_k-C_2 a_k)\upsilon_k'(y)=g'(y),\quad \sum B_k u_k'(y)=0,\quad |y|\leqslant 1$$

The unknown B_k are found from (7).

The solution of the second boundary-value problem for a half-strip with boundary conditions at the end

$$u(0,y)=f(y),\quad \upsilon(0,y)=0$$

is constructed in the same way and can be represented in the form

$$u(x,y)=\sum T_k u_k(y)\exp(i\lambda_k x),\quad \upsilon(x,y)=\sum T_k\upsilon_k(y)\exp(i\lambda_k x),\quad x\geqslant 0$$

$$\frac{\sigma_x(x,y)}{2G}=\begin{cases}\sum T_k\sigma_k(y)\exp(i\lambda_k x), & x>0\\ K_1 y(1-y^2)^{-\gamma}+\sum A_k\sigma_k(y), & x=0\end{cases}$$

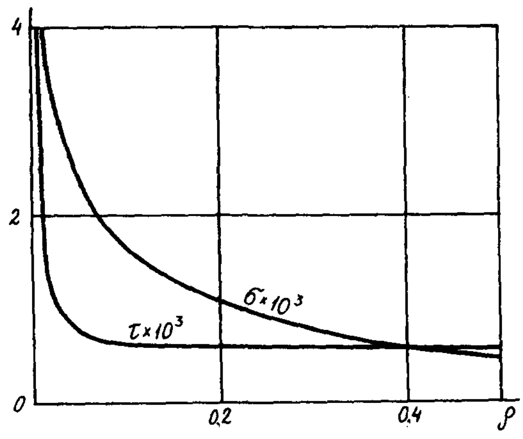

Fig. 1.

$$\frac{\tau_{xy}(x,y)}{2G}=\begin{cases}\sum T_k\tau_k(y)\exp(i\lambda_k x), & x>0\\ K_2(1-y^2)^{-\gamma}+\sum(A_k+K_1a_k-K_2b_k)\tau_k(y), & x=0\end{cases}$$

The coefficients T_k are found from the system

$$\sum a_{nk}T_k=(f,\sigma_n),\quad n=1,2,\ldots$$

where T_k and A_k are linked by the relation $T_k = K_1a_k + A_k$, where K_1 and K_2 are found from the conditions

$$\sum(A_k+K_1a_k-K_2b_k)u_k'(y)=f'(y),\quad \sum A_k v'_k(y)=0,\quad |y|\leqslant 1$$

The solution of the initial boundary-value problem is the sum of the solutions obtained.

A numerical analysis was carried out for a half-strip of material with $\nu = 0.29$. A kinematic load $f(y) = \varepsilon y$, $g(y) = \varepsilon$, where $\varepsilon = 10^{-3}$, is specified at the end. Two frequencies of the exciting load were investigated: $\Omega = 0.5$ and $\Omega = 4$, where $\Omega = 2\varkappa_2/\pi$.

The accuracy with which the boundary conditions δ are satisfied depends on the order of the reduced system N and on the frequency Ω. In the frequency band investigated, an accuracy sufficient for practical purposes ($\delta \leqslant 1.3\%$) is obtained for relatively small values of N ($N = 38$).

At a frequency $\Omega = 0.5$ there is a single travelling wave, the modulus of whose amplitude $|R| = 0.813$, and for $\Omega = 4$ there are four travelling waves with $|R_i| = 0.011, 0.009, 0.848$ and 0.120, respectively. The sharp increase in the efficiency of excitation of the third mode at the frequency $\Omega = 4$ is due to the closeness of the specified load to the form of the wave field of this mode.

The nature of the increase in the stress moduli $\sigma = |\sigma_x|/(2G)$ and $\tau = |\tau_{xy}|/(2G)$ as one approaches a corner point inside the region at an angle of 45° to the end surface is shown in Fig. 1 for a frequency $\Omega = 0.5$.

This research was carried out with financial support from the Russian Foundation for Basic Research (94-01-00220).

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Translated by R.C.G.